

Dilution, decorrelation and scaling in radial growth

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Abstract

The dynamics of fluctuating radially growing interfaces is approached using the formalism of stochastic growth equations on growing domains. This framework reveals a number of dynamic features arising during surface growth. For fast growth, dilution, which spatially reorders the incoming matter, is responsible for the transmission of correlations. Its effects include the erasing of memory with respect to the initial condition, a partial regularization of geometrically originated instabilities, and the restoring of universality in some special cases in which the critical exponents depend on the parameters of the equation of motion. All together lie in the basis of the preservation of the Family-Vicsek scaling in radial interfaces, which is thus a direct consequence of dilution. This fast growth regime is also characterized by the spatial decorrelation of the interface, which in the case of radially growing interfaces naturally originates rapid roughening and polyfractality, and suggests the advent of a self-similar fractal dimension. The center of mass fluctuations of growing clusters are also studied, and our analysis supports a strong violation of the Family-Vicsek scaling by the surface fluctuations of the Eden model. Consequently, this model would belong to a dilution-free universality class.

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I. INTRODUCTION

The study of fluctuating interfaces has occupied an important place within statistical mechanics in recent and not so recent times. The origins of this interest are practical, due to the vast range of potential applications that this theory may have, and theoretical, as some of the universality classes discovered within this framework are claimed to play an important role in other areas of physics [1]. While the great majority of works on this topic has concentrated on strip or slab geometries, it is true that at the very beginning of the theoretical studies on nonequilibrium growth one finds the seminal works by Eden, focused on radial shapes [2, 3]. To a certain extent, the motivation of considering radial forms is related to biological growth, as for instance the Eden model can be thought of as a simplified description of a developing cell colony. The Eden and other related discrete models have been computationally analyzed along the years, and the results obtained has been put in the context of stochastic growth theory, see for instance [4] and references therein.

Apart from the interest in modelling, there is a genuine theoretical motivation in understanding the dynamics of growing radial clusters. The Eden model is actually a sort of first passage percolation [5], and the scaling limit of percolation models has been studied by means of field-theoretic approaches [6] and stochastic processes like Schramm-Loewner evolution [7]. A natural theoretical question to be answered is in which cases the Family-Vicsek scaling [8], basic to describe planar growth processes, is able to capture the behavior of the surface fluctuations of growing radial clusters.

The use of stochastic differential equations, very much spread in the modelling of planar growth profiles, has been not so commonly employed in the case of radial growth. A series of works constitute an exception to this rule [9, 10, 11, 12, 13, 14, 15], as they proposed a partial differential equation with stochastic terms as a benchmark for analyzing the dynamics of radial interfaces. Because studying this sort of equations is complicated by the nonlinearities implied by reparametrization invariance, a simplified version in which only the substrate growth was considered was introduced in [16]. Already in this case it was apparent that for rapidly growing interfaces dilution, which is responsible for matter redistribution as the substrate grows [17], propagates the correlations when large spatiotemporal scales are considered. It is also capable of erasing the memory effects that would otherwise arise, let us show how. In [16] we considered the linear equation for stochastic growth on a growing

domain

$$\partial_t h = -D \left(\frac{t_0}{t} \right)^{\zeta\gamma} |\nabla|^\zeta h - \frac{d\gamma}{t} h + \gamma F t^{\gamma-1} + \left(\frac{t_0}{t} \right)^{d\gamma/2} \xi(x, t), \quad (1)$$

where the domain grows following the power law t^γ , $\gamma > 0$ is the growth index and $-(d\gamma/t)h$ is the term taking into account dilution [16]. Its Fourier transformed version, for $n \geq 1$, is

$$\frac{dh_n}{dt} = -D \left(\frac{t_0}{t} \right)^{\zeta\gamma} \frac{\pi^\zeta |n|^\zeta}{L_0^\zeta} h_n - \frac{d\gamma}{t} h_n + \left(\frac{t_0}{t} \right)^{d\gamma/2} \xi_n(t). \quad (2)$$

This equation can be readily solved for $\gamma > 1/\zeta$ and in the long time limit

$$h_n(t) = (t/t_0)^{-d\gamma} \exp \left[\frac{Dt_0}{1 - \zeta\gamma} \frac{\pi^\zeta |n|^\zeta}{L_0^\zeta} \right] h_n(t_0) + (t/t_0)^{-d\gamma} \int_{t_0}^t \left(\frac{\tau}{t_0} \right)^{d\gamma/2} \xi_n(\tau) d\tau, \quad (3)$$

and so the dependence on the initial condition tends to zero as a power law for long times. This is, as mentioned, one of the consequences of dilution. If we considered the dilation transformation $x \rightarrow (t/t_0)^\gamma x$ we would find again Eq. (1) but this time without the dilution term. The solution now becomes

$$h_n(t) = \exp \left[\frac{Dt_0}{1 - \zeta\gamma} \frac{\pi^\zeta |n|^\zeta}{L_0^\zeta} \right] h_n(t_0) + \int_{t_0}^t \left(\frac{t_0}{\tau} \right)^{d\gamma/2} \xi_n(\tau) d\tau, \quad (4)$$

and so the dependence on the initial condition remains for all times. In the first case the long time solution becomes spatially uncorrelated, and in the second one only part of the initial correlations survive. As an abuse of language, we will talk about decorrelation in both cases. The memory effects that affect the solution in the no-dilution (or dilation) situation separate its behavior from the one dictated by the Family-Vicsek ansatz [16, 18]. For $\gamma < 1/\zeta$ the memory effects and the corresponding dependence on the initial condition disappear exponentially fast for long times as a consequence of the effect of diffusion.

Dilution is also the mechanism that controls the amount of matter on the interface. Pure diffusion on a growing domain is described by the equation

$$\partial_t h = D \left(\frac{t_0}{t} \right)^{2\gamma} \nabla^2 h - \frac{d\gamma}{t} h, \quad (5)$$

in Eulerian coordinates $x \in [0, L_0]$ (see [16]) and where dilution has been taken into account.

The total mass on the surface is conserved

$$\int_0^{L(t)} \cdots \int_0^{L(t)} h(y, t) dy = \left(\frac{t}{t_0} \right)^{d\gamma} \int_0^{L_0} \cdots \int_0^{L_0} h(x, t) dx = \int_0^{L_0} \cdots \int_0^{L_0} h(x, t_0) dx, \quad (6)$$

where $y \equiv [L(t)/L_0]x$ denotes the Lagrangian coordinates. In the no-dilution situation we find

$$\int_0^{L(t)} \cdots \int_0^{L(t)} h(y, t) dy = \left(\frac{t}{t_0}\right)^{d\gamma} \int_0^{L_0} \cdots \int_0^{L_0} h(x, t) dx = \left(\frac{t}{t_0}\right)^{d\gamma} \int_0^{L_0} \cdots \int_0^{L_0} h(x, t_0) dx. \quad (7)$$

This second case is pure dilation, which implies that not only the space grows, but also the interfacial matter grows at the same rate, in such a way that the average density remains constant. Note that this process of matter dilation, as well as the spatial growth, are deterministic processes. These calculations show that both dilution and dilation dynamics are physically motivated and have a number of measurable differences. It is worth remarking here that all previous works except [16] have exclusively considered dilation dynamics. Even in the different field of reaction-diffusion dynamics in which the dilution term was derived, the focus was on the limit in which it was irrelevant [17].

This work is devoted to further explore the consequences of dilution, dilation and decorrelation, and their effects on scaling of radial interfaces. We will use in cases radial stochastic growth equations, which may show up instabilities [15], and explore the interplay of dilution with them. In other cases, when instabilities do not play a determinant role and for the sake of simplicity, we will consider stochastic growth equations on growing domains.

II. RADIAL RANDOM DEPOSITION

In order to construct radial growth equations one may invoke the reparametrization invariance principle [19, 20], as has already been done a number of times [9, 10, 12, 13, 14, 15]. In case of white and Gaussian fluctuations, the d -dimensional spherical noise is given by

$$\frac{1}{\sqrt[4]{g[\vec{\theta}, r(\vec{\theta}, t)]}} \xi(\vec{\theta}, t), \quad \langle \xi(\vec{\theta}, t) \rangle = 0, \quad (8)$$

$$\langle \xi(\vec{\theta}, t) \xi(\vec{\theta}', t) \rangle = \epsilon \delta(\vec{\theta} - \vec{\theta}') \delta(t - t'), \quad (9)$$

where $g = \det(g_{ij}) = \det(\partial_i \vec{r} \cdot \partial_j \vec{r})$ is the determinant of the metric tensor. Under the small gradient assumption $|\nabla_{\vec{\theta}} r| \ll r$ one finds $g \approx \mathcal{J}(r, \vec{\theta})^2$, where \mathcal{J} is the Jacobian determinant of the change of variables from the Cartesian representation (\vec{x}, h) to the polar representation $(\vec{\theta}, r)$. We also have the factorization $\mathcal{J}(r, \vec{\theta})^2 = r^{2d} J(\vec{\theta})^2$, where J is the Jacobian evaluated at $r = 1$.

The simplest growth process is possibly the radial random deposition model. If the growth rate is explicitly time dependent, then the growth equation reads

$$\partial_t r = F\gamma t^{\gamma-1} + \frac{1}{r^{d/2}J(\vec{\theta})^{1/2}}\xi(\vec{\theta}, t), \quad (10)$$

in the absence of dilution. Here $r(\vec{\theta}, t)$ is the radius value at the angular position $\vec{\theta}$ and time t , $F > 0$ is the growth rate, $\gamma > 0$ is the growth index, d is the spatial dimension and ξ is a zero mean Gaussian noise, which correlation is given by

$$\langle \xi(\vec{\theta}, t)\xi(\vec{\theta}', s) \rangle = \epsilon\delta(\vec{\theta} - \vec{\theta}')\delta(t - s). \quad (11)$$

The equation for the first moment can be easily obtained

$$\partial_t \langle r \rangle = F\gamma t^{\gamma-1}, \quad (12)$$

due to the Itô interpretation, and integrate it to get

$$\langle r(\vec{\theta}, t) \rangle = Ft^\gamma, \quad (13)$$

where we have assumed the radially symmetric initial condition $r(\vec{\theta}, t_0) = Ft_0^\gamma$ and $t_0 \leq t$ is the absolute origin of time. It is difficult to obtain more information from the full equation (10), so we will perform a perturbative expansion. We assume the solution form

$$r(\vec{\theta}, t) = R(t) + \sqrt{\epsilon}\rho_1(\vec{\theta}, t), \quad (14)$$

where the noise intensity ϵ will be used as the small parameter [21]. Substituting this solution form into Eq. (10) we obtain the equations

$$\partial_t R = F\gamma t^{\gamma-1}, \quad (15)$$

$$\partial_t \rho_1 = \frac{1}{F^{d/2}t^{\gamma d/2}} \frac{\eta(\vec{\theta}, t)}{J(\vec{\theta})^{1/2}}, \quad (16)$$

where $\xi = \sqrt{\epsilon}\eta$. These equations have been derived assuming $\sqrt{\epsilon} \ll Ft^\gamma$, a condition much more favorable (the better the larger γ is) than the usual time independent ones supporting small noise expansions [21]. The solution to these equations can be readily computed

$$R(\vec{\theta}, t) = Ft^\gamma, \quad (17)$$

$$\langle \rho_1(\vec{\theta}, t) \rangle = 0, \quad (18)$$

$$\langle \rho_1(\vec{\theta}, t)\rho_1(\vec{\theta}', s) \rangle = \frac{F^{-d}}{1-\gamma d} \left[(\min\{t, s\})^{1-\gamma d} - t_0^{1-\gamma d} \right] \frac{\delta(\vec{\theta} - \vec{\theta}')}{J(\vec{\theta})}, \quad (19)$$

if $\gamma d \neq 1$ and where we have assumed a zero value for the initial perturbation. If $\gamma d = 1$ the correlation becomes

$$\left\langle \rho_1(\vec{\theta}, t) \rho_1(\vec{\theta}', s) \right\rangle = \frac{1}{F^d} \ln \left[\frac{\min\{t, s\}}{t_0} \right] \frac{\delta(\vec{\theta} - \vec{\theta}')}{J(\vec{\theta})}. \quad (20)$$

Here R is a deterministic function and ρ_1 is a zero mean Gaussian stochastic process that is completely determined by the correlations given above. The long time behavior of the correlations, given by the condition $t, s \gg t_0$, is specified by the following two-times and one-time correlation functions

$$\left\langle \rho_1(\vec{\theta}, t) \rho_1(\vec{\theta}', s) \right\rangle = \frac{F^{-d}}{1 - \gamma d} (\min\{t, s\})^{1 - \gamma d} \frac{\delta(\vec{\theta} - \vec{\theta}')}{J(\vec{\theta})}, \quad (21)$$

$$\left\langle \rho_1(\vec{\theta}, t) \rho_1(\vec{\theta}', t) \right\rangle = \frac{F^{-d}}{1 - \gamma d} t^{1 - \gamma d} \frac{\delta(\vec{\theta} - \vec{\theta}')}{J(\vec{\theta})}, \quad (22)$$

if $\gamma d > 1$,

$$\left\langle \rho_1(\vec{\theta}, t) \rho_1(\vec{\theta}', s) \right\rangle = \frac{1}{F^d} \ln(\min\{t, s\}) \frac{\delta(\vec{\theta} - \vec{\theta}')}{J(\vec{\theta})}, \quad (23)$$

$$\left\langle \rho_1(\vec{\theta}, t) \rho_1(\vec{\theta}', t) \right\rangle = \frac{1}{F^d} \ln(t) \frac{\delta(\vec{\theta} - \vec{\theta}')}{J(\vec{\theta})}, \quad (24)$$

if $\gamma d = 1$, and finally

$$\left\langle \rho_1(\vec{\theta}, t) \rho_1(\vec{\theta}', s) \right\rangle = \frac{F^{-d}}{\gamma d - 1} t_0^{1 - \gamma d} \frac{\delta(\vec{\theta} - \vec{\theta}')}{J(\vec{\theta})}, \quad (25)$$

when $\gamma d > 1$. In this last case the correlation vanishes in the limit $t_0 \rightarrow \infty$. Note that the reparametrization invariance principle is not able to capture dilution effects and it reproduces pure dilation dynamics.

In order to introduce dilution in the radial case we may use the following functional definition which transforms Eq. (10) into

$$\partial_t r = F \gamma t^{\gamma - 1} - \frac{\gamma d}{t} r + \frac{1}{r^{d/2}} \frac{\xi(\vec{\theta}, t)}{J(\vec{\theta})^{1/2}}, \quad (26)$$

whose first moment can be exactly calculated again taking advantage of the Itô interpretation of the noise term, yielding

$$\left\langle r(\vec{\theta}, t) \right\rangle = \frac{F}{d + 1} t^\gamma. \quad (27)$$

Performing as in the former case the small noise expansion $r = R + \sqrt{\epsilon}\rho_1$ we find again $R = \langle r \rangle$. The perturbation obeys the equation

$$\partial_t \rho_1 = -\frac{\gamma d}{t} \rho_1 + \frac{(d+1)^{d/2}}{F^{d/2} t^{\gamma d/2}} \frac{\eta(\vec{\theta}, t)}{J(\vec{\theta})^{1/2}}, \quad (28)$$

and so the perturbation has zero mean and its long time correlation is given by

$$\left\langle \rho_1(\vec{\theta}, t) \rho_1(\vec{\theta}', s) \right\rangle = \frac{(d+1)^d}{F^d (\gamma d + 1)} \min\{s, t\} \max\{s, t\}^{-\gamma d} \frac{\delta(\vec{\theta} - \vec{\theta}')}{J(\vec{\theta})}, \quad (29)$$

a result that holds uniformly in γ . Note that the structure of the temporal correlation is different when the effect of dilution is considered and when it is not for all $\gamma > 0$. For instance, the characteristic length scale corresponding to a given angular difference is $\lambda = \max\{s, t\}^\gamma |\vec{\theta} - \vec{\theta}'|$ when dilution is present, and $\lambda = \min\{s, t\}^\gamma |\vec{\theta} - \vec{\theta}'|$ in the absence of dilution. One already sees in this example that the lack of dilution causes the appearance of memory effects on the growth dynamics. The first order correction in the small noise expansion ρ_1 is always a Gaussian stochastic process; an attempt to go beyond Gaussianity by deriving the second order correction is reported in appendix A.

III. RANDOM DEPOSITION AND DIFFUSION

Our next step, in order to approach more complex and realistic growth processes, is to add diffusion to a random deposition equation of growth. This sort of equations may be derived using reparametrization invariance as in [15]. Following this reference and the former section, we perform a small noise expansion and concentrate on the equation for the Gaussian perturbation. In this section we will consider a number of cases which do not show instabilities, and the study of these will be postponed to the next one. The equation for the perturbation in $d = 1$ is [15]

$$\partial_t \rho = \frac{D_\zeta}{(F t^\gamma)^\zeta} \Lambda_\theta^\zeta \rho + \frac{1}{\sqrt{F t^\gamma}} \eta(\theta, t), \quad (30)$$

where Λ_θ^ζ is a fractional differential operator of order ζ , and dilution has not been considered. The dynamics for $\zeta > d$, which in turn implies that in the linear case the growth exponent $\beta > 0$ and the interface is consequently rough, has been already considered in [16]; herein we move to studying the marginal case $\zeta = d$, which turns out to have interesting properties.

The case $\zeta < d$ is not so interesting as it corresponds to flat interfaces; an analogous calculation to the corresponding one in [15] for $\gamma = 1$ and $\zeta < 1$ shows

$$\langle \rho(\theta, t) \rho(\theta', s) \rangle \rightarrow 0 \quad \text{when} \quad t, s \rightarrow \infty, \quad (31)$$

independently of the value of t_0 .

If $\zeta = \gamma = 1$ the correlation reads

$$\langle \rho(\theta, t) \rho(\theta', s) \rangle = \frac{1}{4\pi D} \ln \left[\frac{(ts)^{D/F}}{(s/t)^{D/F} + (t/s)^{D/F} - 2 \cos(\theta - \theta')} \right]. \quad (32)$$

The one time correlation adopts the form

$$\langle \rho(\theta, t) \rho(\theta', t) \rangle = \frac{1}{4\pi D} \ln \left[\frac{t^{2D/F}}{2 - 2 \cos(\theta - \theta')} \right], \quad (33)$$

that reduces to

$$\langle \rho(\theta, t) \rho(\theta', t) \rangle \approx \frac{1}{2\pi F} \ln \left(\frac{t}{|\theta - \theta'|^{F/D}} \right), \quad (34)$$

when we consider local in space dynamics, this is, in the limit $\theta \approx \theta'$. Note that this result allows us to define the local dynamic exponent $z_{loc} = F/D \in (0, \infty)$, which depends continuously on the equation parameters F and D , and is thus nonuniversal, as we noted in [15]. In terms of the arc-length variable $\ell - \ell' = t(\theta - \theta')$ we find

$$\langle \rho(\ell, t) \rho(\ell', t) \rangle \approx \frac{F^{-1} + D^{-1}}{2\pi} \ln \left(\frac{t}{|\ell - \ell'|^{F/(D+F)}} \right), \quad (35)$$

where the dynamical exponent in terms of the arc-length variable $z_\ell = F/(D + F) \in (0, 1)$ is again nonuniversal. If we take into account dilution Eq. (30) transforms to

$$\partial_t \rho = \frac{D}{Ft} \Lambda_\theta \rho - \frac{1}{t} \rho + \frac{1}{\sqrt{Ft}} \eta(\theta, t). \quad (36)$$

The solution has zero mean and its correlation is given by

$$\begin{aligned} \langle \rho(\theta, t) \rho(\theta', s) \rangle &= \frac{\min\{s, t\} / \max\{s, t\}}{4\pi F} + \\ &\frac{(\min\{s, t\} / \max\{s, t\})^{1+D/F}}{2\pi(F + D)} \Re \left\{ e^{i(\theta - \theta')} {}_2F_1 \left[1, 1 + \frac{F}{D}; 2 + \frac{F}{D}; e^{i(\theta - \theta')} \left(\frac{\min\{s, t\}}{\max\{s, t\}} \right)^{D/F} \right] \right\}, \end{aligned} \quad (37)$$

where $\Re(\cdot)$ denotes the real part and ${}_2F_1(\cdot, \cdot; \cdot; \cdot)$ is Gauss hypergeometric function [22]. This correlation, for $s = t$ and for small angular scales $\theta \approx \theta'$, becomes at leading order

$$\langle \rho(\theta, t) \rho(\theta', t) \rangle \approx \frac{-1}{2\pi D} \ln(|\theta - \theta'|), \quad (38)$$

which is time independent, and for the arc-length variable

$$\langle \rho(\ell, t) \rho(\ell', t) \rangle \approx \frac{1}{2\pi D} \ln \left(\frac{t}{|\ell - \ell'|} \right), \quad (39)$$

for which the planar scaling and the universal dynamical exponent $z = 1$ are recovered, see Eq. (C5) in [15]. This is yet another example, this time of a different nature, of how dilution is able to restore the Family-Vicsek ansatz [16, 18].

If $\zeta = 1$ and $\gamma < 1$ we find the following correlation function

$$\begin{aligned} \langle \rho(\theta, t) \rho(\theta', s) \rangle = & \frac{[\min\{t, s\}]^{1-\gamma}}{2\pi F(1-\gamma)} - \frac{1}{4\pi D} \ln \left\{ 1 + \exp \left[-\frac{2D}{F(1-\gamma)} |t^{1-\gamma} - s^{1-\gamma}| \right] \right. \\ & \left. - 2 \exp \left[-\frac{D}{F(1-\gamma)} |t^{1-\gamma} - s^{1-\gamma}| \right] \cos(\theta - \theta') \right\}. \end{aligned} \quad (40)$$

When $t = s$ we get

$$\langle \rho(\theta, t) \rho(\theta', t) \rangle = \frac{t^{1-\gamma}}{2\pi F(1-\gamma)} - \frac{1}{4\pi D} \ln [2 - 2 \cos(\theta - \theta')], \quad (41)$$

and considering local spatial dynamics we arrive at

$$\langle \rho(\theta, t) \rho(\theta', t) \rangle \approx \frac{t^{1-\gamma}}{2\pi F(1-\gamma)} - \frac{1}{2\pi D} \ln(|\theta - \theta'|) = \frac{1}{2\pi F(1-\gamma)} \ln \left[\frac{e^{t^{1-\gamma}}}{|\theta - \theta'|^{F(1-\gamma)/D}} \right], \quad (42)$$

expression that does not allow to define a local dynamic exponent, or alternatively $z_{loc} = 0$ due to the exponentially fast spreading of the correlations. These last three expressions contain two clearly different terms. The first one is the zeroth mode component of the correlation, which does not achieve long time saturation. The second term is the nontrivial stationary part of the correlation generated along the evolution. As can be seen, both spatial and temporal correlations are generated.

When the dilution term is taken into account we find the correlation

$$\begin{aligned} \langle \rho(\theta, t) \rho(\theta', s) \rangle = & \frac{\min\{t, s\} [\max\{t, s\}]^{-\gamma}}{2\pi F(\gamma+1)} - \frac{1}{4\pi D} \ln \left\{ 1 + \exp \left[-\frac{2D}{F(1-\gamma)} |t^{1-\gamma} - s^{1-\gamma}| \right] \right. \\ & \left. - 2 \exp \left[-\frac{D}{F(1-\gamma)} |t^{1-\gamma} - s^{1-\gamma}| \right] \cos(\theta - \theta') \right\}. \end{aligned} \quad (43)$$

When $t = s$ we get

$$\langle \rho(\theta, t) \rho(\theta', t) \rangle = \frac{t^{1-\gamma}}{2\pi F(\gamma+1)} - \frac{1}{4\pi D} \ln [2 - 2 \cos(\theta - \theta')], \quad (44)$$

and considering local spatial dynamics we arrive at

$$\langle \rho(\theta, t) \rho(\theta', t) \rangle \approx \frac{t^{1-\gamma}}{2\pi F(\gamma+1)} - \frac{1}{2\pi D} \ln(|\theta - \theta'|) = \frac{1}{2\pi F(\gamma+1)} \ln \left[\frac{e^{t^{1-\gamma}}}{|\theta - \theta'|^{F(\gamma+1)/D}} \right], \quad (45)$$

and we see that as in the former case, both prefactor and exponent are modified, but the still exponentially fast propagation of correlations implies an effective local dynamical exponent $z_{loc} = 0$. Note that for $\gamma > 1$ a radial random deposition behavior for large spatial scales is recovered.

Now we move onto the two-dimensional setting. As in the one-dimensional case we focus on the marginal situation $d = \zeta = 2$, which leads us to denominate this sort of equations as spherical Edwards-Wilkinson (EW) equations, and $0 < \gamma \leq 1/2$, as greater values of the growth index lead again to decorrelation. The straightforward generalization of Eq. (30) is

$$\partial_t \rho = \frac{K}{(Ft^\gamma)^2} \nabla^2 \rho + \frac{1}{Ft^\gamma \sqrt{\sin(\theta)}} \eta(\theta, \phi, t), \quad (46)$$

where the noise is a Gaussian random variable of zero mean and correlation given by

$$\langle \xi(\theta, \phi, t) \xi(\theta', \phi', s) \rangle = \delta(\theta - \theta') \delta(\phi - \phi') \delta(t - s). \quad (47)$$

In this case, if $\gamma < 1/2$, the random variable ρ is a zero mean Gaussian process whose correlation is given by

$$\begin{aligned} \langle \rho(\theta, \phi, t) \rho(\theta', \phi', s) \rangle &= \frac{[\min(t, s)]^{1-2\gamma}}{4\pi F^2 (1-2\gamma)} + \\ &\sum_{l=1}^{\infty} \sum_{m=-l}^l \frac{(-1)^m}{2K(l+l^2)} \exp \left[-\frac{K(l+l^2)}{F^2(1-2\gamma)} |t^{1-2\gamma} - s^{1-2\gamma}| \right] Y_{-m}^l(\theta, \phi) Y_m^l(\theta', \phi'), \end{aligned} \quad (48)$$

where the expansion has been performed on the spherical harmonics basis $Y_m^l(\theta, \phi)$. If $\gamma = 1/2$ then ρ becomes a zero mean Gaussian random variable with the new correlation

$$\begin{aligned} \langle \rho(\theta, \phi, t) \rho(\theta', \phi', s) \rangle &= \frac{\ln[\min(t, s)]}{4\pi F^2} + \\ &\sum_{l=1}^{\infty} \sum_{m=-l}^l \frac{(-1)^m}{2K(l+l^2)} \left[\frac{\min(s, t)}{\max(s, t)} \right]^{K(l+l^2)/F^2} Y_{-m}^l(\theta, \phi) Y_m^l(\theta', \phi'). \end{aligned} \quad (49)$$

It is clear that these correlations are again composed of two different terms, the first one associated with the $l = 0$ mode never saturates, and the second one associated with the rest of modes $l > 0$, which saturates and is responsible of a non-trivial spatial structure.

Taking into account dilution we find for $\gamma < 1/2$ the correlation

$$\begin{aligned} \langle \rho(\theta, \phi, t) \rho(\theta', \phi', s) \rangle &= \frac{\min(t, s) [\max(t, s)]^{-2\gamma}}{4\pi F^2 (2\gamma + 1)} + \\ &\sum_{l=1}^{\infty} \sum_{m=-l}^l \frac{(-1)^m}{2K(l+l^2)} \exp \left[-\frac{K(l+l^2)}{F^2(1-2\gamma)} |t^{1-2\gamma} - s^{1-2\gamma}| \right] Y_{-m}^l(\theta, \phi) Y_m^l(\theta', \phi'), \end{aligned} \quad (50)$$

and for $\gamma = 1/2$

$$\langle \rho(\theta, \phi, t) \rho(\theta', \phi', s) \rangle = \sum_{l=0}^{\infty} \sum_{m=-l}^l \frac{(-1)^m}{2F^2 + 2K(l^2 + l)} \left[\frac{\min(s, t)}{\max(s, t)} \right]^{1+K(l^2+l)/F^2} Y_{-m}^l(\theta, \phi) Y_m^l(\theta', \phi'). \quad (51)$$

In the two dimensional situation we see that dilution also has a measurable effect, which is more pronounced in the critical $\gamma = 1/2$ case. For this value all the modes in the correlation saturate and contribute to create a stationary spatial structure, as in the one-dimensional setting. It is difficult to establish more comparisons among both dimensionalities, as the infinite sums that were explicit in $d = 1$ become much more involved in $d = 2$, due to the double series containing the spherical harmonics. We however conjecture that the modification of the scaling properties due to effect of dilution in two dimensions is similar to the one explicitly observed in one dimension.

IV. INSTABILITIES

A spherical EW equation derived from the geometric principle of surface minimization was introduced in [15]. The corresponding equation for the radius $r(\theta, \phi, t)$ reads

$$\partial_t r = K \left[\frac{\partial_\theta r}{r^2 \tan(\theta)} + \frac{\partial_\theta^2 r}{r^2} + \frac{\partial_\phi^2 r}{r^2 \sin^2(\theta)} - \frac{2}{r} \right] + F\gamma t^{\gamma-1} + \frac{1}{r\sqrt{\sin(\theta)}} \xi(\theta, \phi, t). \quad (52)$$

Performing the small noise expansion $r(\theta, \phi, t) = Ft + \rho(\theta, \phi, t)$ we find a linear equation which differs from Eq. (46) in that it has a destabilizing term coming from the fourth term in the drift of Eq. (52), see [15]. In this reference one can see that in the absence of dilution the $l = 0$ mode is unstable and the $l = 1$ modes are marginal while the rest of modes is stable. The effect of this sort of geometrically originated instability on the mean value of the stochastic perturbation and alternative geometric variational approaches that avoid it can be seen in [15], herein we will concentrate on its effect on correlations. Its effect on mean values can be easily deduced from them.

In the long time limit and provided $\gamma < 1/2$, the perturbation is a Gaussian process

whose correlation is given by

$$\begin{aligned} \langle \rho(\theta, \phi, t) \rho(\theta', \phi', s) \rangle &= \frac{1}{16\pi K} \exp \left[\frac{2K(t^{1-2\gamma} + s^{1-2\gamma})}{F^2(1-2\gamma)} \right] + \\ &\frac{3 [\min(t, s)]^{1-2\gamma}}{4\pi F^2(1-2\gamma)} [\cos(\theta) \cos(\theta') + \cos(\phi - \phi') \sin(\theta) \sin(\theta')] + \\ &\sum_{l=2}^{\infty} \sum_{m=-l}^l \frac{(-1)^m}{2K(l^2 + l - 2)} \exp \left[-\frac{K(l^2 + l - 2)}{F^2(1-2\gamma)} |t^{1-2\gamma} - s^{1-2\gamma}| \right] Y_{-m}^l(\theta, \phi) Y_m^l(\theta', \phi'). \end{aligned} \quad (53)$$

If $\gamma = 1/2$ the correlation shifts to

$$\begin{aligned} \langle \rho(\theta, \phi, t) \rho(\theta', \phi', s) \rangle &= \frac{(st/t_0^2)^{2K/F^2}}{16\pi K} + \\ &\frac{3 \ln [\min(s, t)]}{4\pi F^2} [\cos(\theta) \cos(\theta') + \cos(\phi - \phi') \sin(\theta) \sin(\theta')] + \\ &\sum_{l=2}^{\infty} \sum_{m=-l}^l \frac{(-1)^m}{2K(l^2 + l - 2)} \left[\frac{\min(s, t)}{\max(s, t)} \right]^{K(l^2 + l - 2)/F^2} Y_{-m}^l(\theta, \phi) Y_m^l(\theta', \phi'). \end{aligned} \quad (54)$$

In these cases the modes characterized by $l = 0$ and $l = 1$ do not saturate, and the rest of the modes $l > 1$ saturate and create a non-trivial spatial structure. When $\gamma < 1/2$ the $l = 1$ modes grow in time as a power law with the exponent $1 - 2\gamma$, while the $l = 0$ mode grows exponentially fast. When $\gamma = 1/2$ the $l = 1$ modes grow logarithmically and the $l = 0$ mode grows as a power law with the non-universal exponent $4K/F^2$.

When we consider the effect of dilution, and for $\gamma < 1/2$, we find the correlation

$$\begin{aligned} \langle \rho(\theta, \phi, t) \rho(\theta', \phi', s) \rangle &= \frac{1}{16\pi K} \exp \left[\frac{2K(t^{1-2\gamma} + s^{1-2\gamma})}{F^2(1-2\gamma)} \right] + \\ &\frac{3 \min(t, s) [\max(t, s)]^{-2\gamma}}{4\pi F^2(2\gamma + 1)} [\cos(\theta) \cos(\theta') + \cos(\phi - \phi') \sin(\theta) \sin(\theta')] + \\ &\sum_{l=2}^{\infty} \sum_{m=-l}^l \frac{(-1)^m}{2K(l^2 + l - 2)} \exp \left[-\frac{K(l^2 + l - 2)}{F^2(1-2\gamma)} |t^{1-2\gamma} - s^{1-2\gamma}| \right] Y_{-m}^l(\theta, \phi) Y_m^l(\theta', \phi'). \end{aligned} \quad (55)$$

For $\gamma = 1/2$ the correlation reads

$$\begin{aligned} \langle \rho(\theta, \phi, t) \rho(\theta', \phi', s) \rangle &= \frac{1}{4\pi} \langle \rho_0^0(t) \rho_0^0(s) \rangle + \\ &\sum_{l=1}^{\infty} \sum_{m=-l}^l \frac{(-1)^m}{2F^2 + 2K(l^2 + l - 2)} \left[\frac{\min(s, t)}{\max(s, t)} \right]^{1+K(l^2 + l - 2)/F^2} Y_{-m}^l(\theta, \phi) Y_m^l(\theta', \phi'), \end{aligned} \quad (56)$$

where

$$\langle \rho_0^0(t) \rho_0^0(s) \rangle = \begin{cases} (2F^2 - 4K)^{-1} (\min\{s, t\} / \max\{s, t\})^{1-2K/F^2} & \text{if } F^2 > 2K, \\ \ln(\min\{s, t\}) / F^2 & \text{if } F^2 = 2K, \\ (4K - 2F^2)^{-1} (ts/t_0^2)^{2K/F^2-1} & \text{if } F^2 < 2K, \end{cases} \quad (57)$$

where t_0 is the absolute origin of time.

Contrary to what happens in the stable case, Eq. (46), in the unstable case with no dilution, Eq. (52), the $l = 0$ mode is unstable, showing an exponential growth, and the $l = 1$ modes shows an algebraic increase with the universal exponent $1 - 2\gamma$, provided $\gamma < 1/2$; the rest of modes is stable. The marginal value of the growth index $\gamma = 1/2$ translates into a power law increase of the $l = 0$ mode with a non-universal exponent, while the $l = 1$ modes grow logarithmically; the rest of modes is again stable. It is clear that dilution has a stabilizing effect. Indeed, for $\gamma < 1/2$ the $l = 0$ mode is unchanged, but the $l = 1$ modes, which still grow in time, experience a lost of memory effects. In the critical $\gamma = 1/2$ situation the dilution effects are stronger. The $l = 1$ modes, which formerly grew logarithmically, now become stable; the $l = 0$ mode, which formerly showed an algebraic growth, now shows (non-universal) algebraic or logarithmic grow, or even saturation, depending on the relation among the values of the parameters of the spherical EW equation. In any case, even that of algebraic growth, this growth is always slower than in the no dilution situation. Stable modes saturate contributing to create a non-trivial spatial structure in the whole range $\gamma \leq 1/2$.

In summary, the effect of dilution is weakly stabilizing in the subcritical case, while stronger and more identifiable in criticality. Of course, the supercritical situation is characterized by an effective random deposition behavior in the large spatial scale.

V. INTRINSICALLY SPHERICAL GROWTH AND RAPID ROUGHENING

It is necessary to clarify the role of the diffusivity index ζ . We have defined it as the order of the fractional differential operator taking mass diffusion into account, and so far we have referred to it as the key element triggering decorrelation. This has been an abuse of language because we have assumed that the negative power of the radius (or its mean field analog Ft^γ – what really matters is the resulting power of the temporal variable) preceding this differential operator was exactly ζ . This would not be the case if the diffusion constant were time or radius dependent, but also in some other cases as the Intrinsically Spherical (IS) equation derived from geometric variational principles in [15]. This equation was obtained as a gradient flow pursuing the minimization of the interface mean curvature, and then linearizing with respect to the different derivatives of the radius as given by the small gradient

assumption [15]. It is termed “intrinsically spherical” because it has no planar counterpart, as the nonlinearity becomes fundamental in any attempt to derive such a gradient flow in the Cartesian framework [23]. Note the similarity of this with other equations classical in this context, as the EW equation is a gradient flow which minimizes the surface area and the Mullins-Herring equation minimizes the interface square mean curvature [15]; the IS equation, as mentioned, minimizes the interface mean curvature. It reads [15]

$$\partial_t r = K \left[\frac{\partial_\theta^2 r}{r^3} + \frac{\partial_\phi^2 r}{r^3 \sin^2(\theta)} + \frac{\partial_\theta r}{r^3 \tan(\theta)} - \frac{1}{r^2} \right] + F\gamma t^{\gamma-1} + \frac{1}{r\sqrt{\sin(\theta)}} \xi(\theta, \phi, t), \quad (58)$$

and so $\zeta = 2$ in this case, but however one finds a factor r^{-3} in front of the diffusive differential operator, instead of the r^{-2} factor characteristic of the EW equation. This difference will have a number of measurable consequences, as we will show in the following. The equation for the stochastic perturbation reads in this case

$$\frac{d\rho_m^l}{dt} = \frac{K}{F^3 t^{3\gamma}} [2 - l(l+1)] \rho_m^l - \frac{2\gamma}{t} \rho_m^l + \frac{1}{F t^\gamma} \eta_m^l(t), \quad (59)$$

which reveals that the critical value of the growth index $\gamma = 1/3$; a faster growth leads to decorrelation. This is the first but not the unique difference with respect to the EW equation. To find out more we will first put things in a broader context.

A more general equation for radial growth, after introducing dilution, would be

$$\partial_t r = -\frac{K}{r^\delta} |\nabla|^\zeta r - \frac{\gamma d}{t} r + F\gamma t^{\gamma-1} + \frac{\sqrt{\epsilon}}{\sqrt{r^d J(\vec{\theta})}} \eta(\vec{\theta}, t), \quad (60)$$

which defines the damping index δ , differing from the diffusivity index ζ in general; note that Eq. (60) has left aside the instability properties of the IS equation, which are analogous to those of the EW equation, and would add nothing to last section discussion. For simplicity we will focus on values of the damping index fulfilling $\delta \geq \zeta$. This equation can be treated perturbatively for small ϵ following the previous sections procedure and by introducing the hyperspherical harmonics $Y_l^{\vec{m}}(\vec{\theta})$, which obey the eigenvalue equation [24]

$$\nabla^2 Y_l^{\vec{m}}(\vec{\theta}) = -l(l+d-1) Y_l^{\vec{m}}(\vec{\theta}), \quad (61)$$

where the vector \vec{m} represents the set of $(d-1)$ indices. The fractional operator acts on the hyperspherical harmonics in the following fashion

$$|\nabla|^\zeta Y_l^{\vec{m}}(\vec{\theta}) = [l(l+d-1)]^{\zeta/2} Y_l^{\vec{m}}(\vec{\theta}). \quad (62)$$

The hyperspherical noise is Gaussian, has zero mean and its correlation is given by

$$\langle \eta(\vec{\theta}, t) \eta(\vec{\theta}', t') \rangle = \delta(\vec{\theta} - \vec{\theta}') \delta(t - t'). \quad (63)$$

It can be expanded in terms of hyperspherical harmonics

$$\frac{\eta(\vec{\theta}, t)}{\sqrt{J(\vec{\theta})}} = \sum_{l, \vec{m}} \eta_l^{\vec{m}}(t) Y_l^{\vec{m}}(\vec{\theta}), \quad (64)$$

and the amplitudes are given by

$$\eta_l^{\vec{m}}(t) = \int \eta(\vec{\theta}, t) \bar{Y}_l^{\vec{m}}(\vec{\theta}) \sqrt{J(\vec{\theta})} d\vec{\theta}, \quad (65)$$

and so they are zero mean Gaussian noises whose correlation is given by

$$\langle \eta_l^{\vec{m}}(t) \bar{\eta}_{l'}^{\vec{m}'}(t') \rangle = \delta(t - t') \delta_{l, l'} \delta_{\vec{m}, \vec{m}'}, \quad (66)$$

where the overbar denotes complex conjugation. Note that the amplitudes are in general complex valued. They obey the linear equation

$$\frac{d\rho_l^{\vec{m}}}{dt} = -\frac{K}{F\delta t^{\delta\gamma}} [l(l+d-1)]^{\zeta/2} \rho_l^{\vec{m}} - \frac{\gamma d}{t} \rho_l^{\vec{m}} + \frac{1}{F^{d/2} t^{\gamma d/2}} \eta_l^{\vec{m}}(t). \quad (67)$$

From this equation it is clear that the critical value of the growth index is $\gamma = 1/\delta$, and a faster growth leads to decorrelation.

It is convenient to move to a growing hypercubic geometry as in [16] in order to calculate different quantities

$$\partial_t h = -D \left(\frac{t_0}{t} \right)^{\delta\gamma} |\nabla|^\zeta h - \frac{d\gamma}{t} h + \gamma F t^{\gamma-1} + \left(\frac{t_0}{t} \right)^{d\gamma/2} \xi(x, t), \quad (68)$$

since this change simplifies calculations without modifying the leading results. Our goal is finding the growth and auto-correlation exponents, as this last one is a good quantity to measure decorrelation [16]. In order to calculate the temporal correlations we need to consider the short time limit, where the growth exponent β becomes apparent. The propagator of Eq.(68) is

$$G_n(t) = \left(\frac{t}{t_0} \right)^{-d\gamma} \exp \left[-\frac{n^\zeta \pi^\zeta D t_0^{\gamma\delta} t^{1-\gamma\delta} - t_0}{L_0^\zeta (1 - \gamma\delta)} \right], \quad (69)$$

that yields the following complete solution when the initial condition vanishes:

$$h_n(t) = G_n(t) \int_{t_0}^t G_n^{-1}(\tau) \left(\frac{t_0}{\tau} \right)^{d\gamma/2} \xi_n(\tau) d\tau. \quad (70)$$

The one point two times correlation function then reads

$$\langle h_n(t)h_n(t') \rangle \sim G_n(t)G_n(t') \int_{t_0}^{\min(t,t')} G_n^{-2}(\tau) \left(\frac{t_0}{\tau}\right)^{d\gamma} d\tau, \quad (71)$$

and after inverting Fourier we arrive at the real space expression

$$\langle h(x,t)h(x,t') \rangle = \sum_{n=0}^{\infty} \langle h_n(t)h_n(t') \rangle \cos^2\left(\frac{n\pi x}{L_0}\right), \quad (72)$$

where we have assumed no flux boundary conditions as in [16], although the values of both the growth and auto-correlation exponents do not depend on the choice of boundary conditions. The propagator $G_n(t)$ suggests the scaling variable $v_n \sim nt^{(1-\gamma\delta)/\zeta}$ in Fourier space, that corresponds to the real space scaling variable $u \sim xt^{(-1+\gamma\delta)/\zeta}$, as can be read directly from Eq. (72). This suggests the definition of the effective dynamical exponent $z_{\text{eff}} = \zeta/(1 - \gamma\delta)$. If we express the correlation Eq. (71) for $t = t'$ in terms of the scaling variable v_n (and we refer to it as $C(v_n)$ multiplied by a suitable power of t) and we introduce the “differential” $1 \equiv \Delta n \sim t^{(-1+\gamma\delta)/\zeta} \Delta v$, we can cast the last expression in the integral form

$$\langle h(x,t)^2 \rangle - \langle h(x,t) \rangle^2 = t^{1-d/\zeta+\gamma d(\delta/\zeta-1)} \int_{v_1}^{\infty} C(v_n) \cos^2\left(\frac{v_n \pi u}{L_0}\right) dv_n, \quad (73)$$

where the series converges as a Riemann sum to the above integral when

$$Dt \ll (L_0^\zeta + Dt_0) \frac{t^{\delta\gamma}}{t_0^{\delta\gamma}}, \quad (74)$$

or equivalently $t \ll t_c \sim L_0^{z_{\text{eff}}}$, for t_c being the time it takes the correlations reaching the substrate boundaries, assuming that the substrate initial size is very large. If $\gamma < 1/\delta$, the whole substrate becomes correlated, yielding a finite t_c ; for $\gamma > 1/\delta$ the convergence of the Riemann sum to the integral is assured for all times, corresponding to the physical fact that the substrate never becomes correlated. In front of the integral we find a power of the temporal variable compatible with the growth exponent

$$\beta = \frac{1}{2} - \frac{d}{2\zeta} + \frac{\gamma d}{2} \left(\frac{\delta}{\zeta} - 1\right), \quad (75)$$

and the integral can be shown to be absolutely convergent as the integrand decays faster than exponentially for large values of the scaling variable v_n .

We are now in position to calculate the temporal auto-correlation

$$A(t,t') \equiv \frac{\langle h(x,t)h(x,t') \rangle_0}{\langle h(x,t)^2 \rangle_0^{1/2} \langle h(x,t')^2 \rangle_0^{1/2}} \sim \left(\frac{\min\{t,t'\}}{\max\{t,t'\}}\right)^\lambda, \quad (76)$$

where λ is the auto-correlation exponent and $\langle \cdot \rangle_0$ denotes the average with the zeroth mode contribution suppressed, as in (73). The remaining ingredient is the correlation $\langle h(x, t)h(x, t') \rangle_0$. Going back to Eq.(72) we see that the Fourier space scaling variable now reads

$$v_n = \left[\frac{t^{1-\gamma\delta} + (t')^{1-\gamma\delta} - 2\tau^{1-\gamma\delta}}{1 - \gamma\delta} \right]^{1/\zeta} n. \quad (77)$$

If $\gamma < 1/\delta$ the term $\max\{t, t'\}^{1-\gamma\delta}$ is dominant and the factor in front of the convergent Riemann sum reads

$$\max\{t, t'\}^{(\delta/\zeta - 1)\gamma d - d/\zeta} \min\{t, t'\}, \quad (78)$$

after the time integration has been performed and in the limit $\max\{t, t'\} \gg \min\{t, t'\}$. In this same limit, but when $\gamma > 1/\delta$, the term $\min\{t, t'\}^{1-\gamma\delta}$ becomes dominant and the prefactor reads

$$\max\{t, t'\}^{-d\gamma} \min\{t, t'\}^{1-d/\zeta + d\gamma\delta/\zeta}. \quad (79)$$

The resulting temporal correlation adopts the form indicated in the right hand side of (76), where

$$\lambda = \begin{cases} \beta + d/\zeta + \gamma d(1 - \delta/\zeta) & \text{if } \gamma < 1/\delta, \\ \beta + \gamma d & \text{if } \gamma > 1/\delta, \end{cases} \quad (80)$$

or alternatively

$$\lambda = \beta + \frac{d}{z_\lambda}, \quad (81)$$

where the λ -dynamical exponent is defined as

$$z_\lambda = \begin{cases} \frac{\zeta}{1 + \gamma(\zeta - \delta)} & \text{if } \gamma < 1/\delta, \\ 1/\gamma & \text{if } \gamma > 1/\delta. \end{cases} \quad (82)$$

If we disregarded the effect of dilution we would find again Eq. (81), but this time

$$z_\lambda = \begin{cases} \frac{\zeta}{1 - \gamma\delta} = z_{\text{eff}} & \text{if } \gamma < 1/\delta, \\ \infty & \text{if } \gamma > 1/\delta. \end{cases} \quad (83)$$

To further clarify the dynamics we now calculate the scaling form that the two points correlation function adopts for short spatial scales $|x - x'| \ll t^{(1-\delta\gamma)/\zeta}$ in the decorrelated regime. As dilution does not act on such a microscopic scale, the following results are independent of whether we contemplate dilution or not. In this case one has

$$\langle h(x, t)h(x', t) \rangle = \sum_{n_1, \dots, n_d} \langle h_n^2(t) \rangle \cos\left(\frac{n_1 \pi x_1}{L_0}\right) \cos\left(\frac{n_1 \pi x'_1}{L_0}\right) \cdots \cos\left(\frac{n_d \pi x_d}{L_0}\right) \cos\left(\frac{n_d \pi x'_d}{L_0}\right), \quad (84)$$

where $x = (x_1, \dots, x_d)$ and $n = (n_1, \dots, n_d)$, and we assume the rough interface inequality $\zeta > d$ in order to assure the absolute convergence of this expression. By introducing the scaling variables $v_i = n_i t^{(1-\delta\gamma)/\zeta}$ and $u_i = x_i t^{(\gamma\delta-1)/\zeta}$ for $i = 1, \dots, d$ and assuming statistical isotropy and homogeneity of the scaling form we find

$$\langle h(x, t)h(x', t) \rangle - \langle h(x, t) \rangle^2 = |x - x'|^{\zeta-d} t^{\gamma(\delta-d)} \mathcal{F} [|x - x'| t^{(\delta\gamma-1)/\zeta}], \quad (85)$$

or in Lagrangian coordinates $|y - y'| = |x - x'| t^\gamma$

$$\langle h(y, t)h(y', t) \rangle - \langle h(y, t) \rangle^2 = |y - y'|^{\zeta-d} t^{\gamma(\delta-\zeta)} \mathcal{F} \left[\frac{|y - y'|}{t^{\{1+\gamma(\zeta-\delta)\}/\zeta}} \right]. \quad (86)$$

We see that this form is statistically self-affine with respect to the re-scaling $y \rightarrow by$, $t \rightarrow b^z t$, and $h \rightarrow b^\alpha h$, where the critical exponents are

$$\alpha = \frac{\zeta - d}{2} + \frac{\zeta}{1 + \gamma(\zeta - \delta)} \frac{(\delta - \zeta)\gamma}{2}, \quad z = \frac{\zeta}{1 + \gamma(\zeta - \delta)}. \quad (87)$$

Note that the scaling relation $\alpha = \beta z$ holds, where the growth exponent β was calculated in Eq. (75). The macroscopic decorrelation, which is observed for length scales of the order of the system size $|x - x'| \approx L_0$, is controlled by the effective dynamical exponent z_{eff} . When $\delta > \zeta$ decorrelation might happen at microscopic length scales $|x - x'| \ll t^{(1-\delta\gamma)/\zeta}$ as well. Microscopic decorrelation happens in the limit $\delta \rightarrow \zeta + 1/\gamma$. For $\delta < \zeta + 1/\gamma$ the interface is microscopically correlated and the critical exponents take on their finite values given in Eq. (87). For $\delta \geq \zeta + 1/\gamma$ the interface is microscopically uncorrelated and the critical exponents diverge $\alpha = z = \infty$, while the growth exponent is still finite and given by Eq. (75) (so one could say the scaling relation $\alpha = \beta z$ still holds in some sense in the microscopic uncorrelated limit). With respect to the growth exponent we can say that $\beta < 1/2$ when $\delta < \gamma^{-1} + \zeta$, $\beta \rightarrow 1/2$ when $\delta \rightarrow \gamma^{-1} + \zeta$, and $\beta > 1/2$ when $\delta > \gamma^{-1} + \zeta$, so rapid roughening is a consequence of microscopic decorrelation. And now, by applying the developed theory to the IS equation, for which $d = 2$, $\zeta = 2$, $\delta = 3$ and assuming as in [15] that $\gamma = 1$, we find that it is exactly positioned at the threshold of microscopic decorrelation, this is, its critical exponents are $\alpha = z = \infty$ and $\beta = 1/2$.

Note that the effective dynamical exponent $z_{\text{eff}} = \zeta/(1 - \gamma\delta)$ states the speed at which both correlation and decorrelation occur. The transition from correlation to decorrelation is triggered by the comparison among the indexes γ and δ . The derivation order ζ controls the speed at which both processes happen: a larger ζ implies slower correlation/decorrelation

processes. Note also that rapid roughening might appear in exactly the same way in planar processes, just by allowing field or time dependence on the diffusion constant. This could be thought as somehow artificial in some planar situations, but as we have shown it appears naturally in the radial case, where such a dependence is a straightforward consequence of the lost of translation invariance, due to the existence of an absolute origin of space, characterized by a zero radius (and which in turn implies the existence of an absolute origin of time in the small noise approximation, as we have already seen). Such a naturalness can be seen in the derivation of the IS equation in [15], where it was found as a consequence of a simple variational principle.

VI. POLYFRACTALITY

We devote this section to showing that rapidly growing radial interfaces develop “polyfractality”. We coin this term to denote a behavior characterized by a scale dependent fractal dimension taking place in a finite system and for long times. It is different from the concept of multifractality, which in this topic is usually associated to a nonlinear relation among the exponents characterizing the higher order height difference correlations [1].

In the classical case of static planar interfaces the fractal dimension is computed from the height difference correlation function

$$\langle [h(x, t) - h(x', t)]^2 \rangle^{1/2} \sim |x - x'|^H, \quad (88)$$

in the long time limit, i. e. after saturation have been achieved, where the Hurst exponent $H = (\zeta - d)/2$ for linear growth equations and the right hand side is time independent. The interface fractional dimension is calculated using the box counting method and is given by $d_f = 1 + d - H$. The general linear equation for stochastic growth on a growing domain was found in the last section to be

$$\partial_t h = -D \left(\frac{t_0}{t} \right)^{\delta\gamma} |\nabla|^\zeta h - \frac{d\gamma}{t} h + \gamma F t^{\gamma-1} + \left(\frac{t_0}{t} \right)^{d\gamma/2} \xi(x, t), \quad (89)$$

for which we will assume $\zeta \leq \delta < \zeta + \gamma^{-1}$. Its Fourier transformed version, for $n \geq 1$, is

$$\frac{dh_n}{dt} = -D \left(\frac{t_0}{t} \right)^{\delta\gamma} \frac{\pi^\zeta |n|^\zeta}{L_0^\zeta} h_n - \frac{d\gamma}{t} h_n + \left(\frac{t_0}{t} \right)^{d\gamma/2} \xi_n(t). \quad (90)$$

For slow growth $\gamma < 1/\delta$ diffusion dominates over dilution and one finds an expression compatible with that of the planar case

$$\langle [h(x, t) - h(x', t)]^2 \rangle^{1/2} \sim t^{\gamma(\delta-d)/2} |x - x'|^{(\zeta-d)/2}, \quad (91)$$

and so the Hurst exponent and interface fractal dimension are the same as in the planar case for fixed time. In the case of fast growth $\gamma > 1/\delta$, for small spatial scales $|x - x'| \ll t^{(1-\delta\gamma)/\zeta}$ we recover again this result, while for large spatial scales $|x - x'| \gg t^{(1-\delta\gamma)/\zeta}$ we find

$$\langle [h(x, t) - h(x', t)]^2 \rangle^{1/2} \sim t^\beta, \quad (92)$$

and so, for fixed time, $H = 0$ and $d_f = d + 1$. This means that the interface becomes highly irregular and so dense that it fills the $(d + 1)$ -dimensional space. This way decorrelation marks the onset of polyfractality, as specified by a scale dependent Hurst exponent, whose asymptotic values are

$$H(|x - x'|, t) = \begin{cases} (\zeta - d)/2 & \text{if } |x - x'| \ll t^{(1-\delta\gamma)/\zeta}, \\ 0 & \text{if } |x - x'| \gg t^{(1-\delta\gamma)/\zeta}, \end{cases} \quad (93)$$

and the corresponding asymptotic values of the scale dependent fractal dimension

$$d_f(|x - x'|, t) = \begin{cases} 1 + (3d - \zeta)/2 & \text{if } |x - x'| \ll t^{(1-\delta\gamma)/\zeta}, \\ d + 1 & \text{if } |x - x'| \gg t^{(1-\delta\gamma)/\zeta}. \end{cases} \quad (94)$$

Note that these results imply dynamic polyfractality as the scale separating the two regimes depends on time $|x - x'| \sim t^{(1-\delta\gamma)/\zeta}$; also, the rough interface inequality $\zeta > d$ implies the strict inequality $1 + (3d - \zeta)/2 < d + 1$. This asymptotic behavior strongly suggests the self-similar form of both Hurst exponent and fractal dimension

$$H = H\left(\frac{|x - x'|}{t^{(1-\delta\gamma)/\zeta}}\right), \quad \text{and} \quad d_f = d_f\left(\frac{|x - x'|}{t^{(1-\delta\gamma)/\zeta}}\right). \quad (95)$$

According to this the fractal dimension would be a dynamic fractal itself, invariant to the transformation $x \rightarrow bx$, $t \rightarrow b^{z_f}t$, and $d_f \rightarrow b^{\alpha_f}d_f$, for $z_f = \zeta/(1 - \delta\gamma) = z_{\text{eff}}$ and $\alpha_f = 0$. Note that all these results concerning polyfractality are independent of whether we contemplate dilution or not (because the height difference correlation function depends on strictly local quantities [16]), and so we could, in this particular calculation, substitute Eqs. (89) and (90) by their dilution-free counterparts and still get the same results. Note

also that at the very beginning of this section we have assumed the inequality $\zeta \leq \delta < \zeta + \gamma^{-1}$, which implies that for rapid growth the interface is macroscopically but not microscopically uncorrelated. If $\delta \geq \zeta + \gamma^{-1}$ then the interface is microscopically uncorrelated and the fractal dimension becomes $d_f = d + 1$ independently of the scale from which we regard it, i. e., polyfractality is a genuine effect of macroscopic decorrelation, which disappears for strong damping causing microscopic decorrelation.

Note that polyfractality does not appear in non-growing domain systems as for long times saturation is achieved and the fractal dimension becomes constant (assuming no multifractality is present). Although the behavior of the height difference correlation function we found here is similar to the one present in classical unbounded systems, results concerning the fractal dimension cannot be immediately extrapolated. The fractal dimension can be computed in a bounded growing domain, for instance using the box counting method as we have done herein, by employing as the reference length $L(t)$, the linear time dependent size of the system. Of course, in an unbounded static domain there is not such a reference length.

VII. THE KARDAR-PARISI-ZHANG EQUATION

One of the most important nonlinear models in the field of surface growth is the Kardar-Parisi-Zhang (KPZ) equation [25]

$$\partial_t h = \nu \nabla^2 h + \lambda (\nabla h)^2 + \xi(x, t). \quad (96)$$

It is related to the biologically motivated Eden model, as this model, at least in a planar geometry, was numerically found to belong to the KPZ universality class [1]. As we will see, understanding the KPZ equation on a growing domain may shed some light on some of the properties of the classical version of this model.

The KPZ equation on a growing domain reads

$$\partial_t h = \nu \left(\frac{t_0}{t}\right)^{2\gamma} \nabla^2 h + \frac{\lambda}{2} \left(\frac{t_0}{t}\right)^{2\gamma} (\nabla h)^2 - \frac{d\gamma}{t} h + \gamma F t^{\gamma-1} + \left(\frac{t_0}{t}\right)^{d\gamma/2} \xi(x, t). \quad (97)$$

Of course, if we just considered the dilation $x \rightarrow (t/t_0)^\gamma x$ we would find

$$\partial_t h = \nu \left(\frac{t_0}{t}\right)^{2\gamma} \nabla^2 h + \frac{\lambda}{2} \left(\frac{t_0}{t}\right)^{2\gamma} (\nabla h)^2 + \gamma F t^{\gamma-1} + \left(\frac{t_0}{t}\right)^{d\gamma/2} \xi(x, t). \quad (98)$$

As we have shown in the previous section, the dilution mechanism fixes the Family-Vicsek ansatz in the fast growth regime. In the radial Eden model case, assuming it belongs to the KPZ universality class, we would have $z = 3/2$ in $d = 1$ and $\gamma = 1$. And so, one would naïvely expect that the resulting interface is uncorrelated and we have to resort on dilution effects in order to fix the Family-Vicsek ansatz and get rid of memory effects. But here comes the paradoxical situation. There are two main symmetries associated with the d -dimensional KPZ equation: the Hopf-Cole transformation which maps it onto the noisy diffusion equation [26] and the related directed polymer problem [27, 28], and Galilean invariance which have been traditionally related to the non-renormalization of the KPZ vertex at an arbitrary order in the perturbation expansion [29, 30]. In the case of the no-dilution KPZ equation (98) both symmetries are still present. Indeed, this equation transforms under the Hopf-Cole transformation $u = \exp[\lambda h/(2\nu)]$ to

$$\partial_t u = \nu \left(\frac{t_0}{t} \right)^{2\gamma} \nabla^2 u + \frac{\gamma F \lambda}{2\nu} t^{\gamma-1} u + \frac{\lambda}{2\nu} \left(\frac{t_0}{t} \right)^{d\gamma/2} \xi(x, t) u, \quad (99)$$

which is again a noisy diffusion equation and it can be explicitly solved in the deterministic limit $\epsilon = 0$. We find in this case

$$u(x, t) = \frac{(1 - 2\gamma)^{d/2} \exp[F\lambda t^\gamma/(2\nu)]}{[4\pi t_0^{2\gamma}(t^{1-2\gamma} - t_0^{1-2\gamma})]^{d/2}} \int_{\mathbb{R}^d} \exp \left[-\frac{|x - y|^2(1 - 2\gamma)}{4t_0^{2\gamma}(t^{1-2\gamma} - t_0^{1-2\gamma})} \right] u(y, t_0) dy, \quad (100)$$

which corresponds to

$$h(x, t) = \frac{2\nu}{\lambda} \ln \left\{ \frac{(1 - 2\gamma)^{d/2} \exp[F\lambda t^\gamma/(2\nu)]}{[4\pi t_0^{2\gamma}(t^{1-2\gamma} - t_0^{1-2\gamma})]^{d/2}} \int_{\mathbb{R}^d} \exp \left[-\frac{|x - y|^2(1 - 2\gamma)}{4t_0^{2\gamma}(t^{1-2\gamma} - t_0^{1-2\gamma})} + \frac{\lambda}{2\nu} h(y, t_0) \right] dy \right\}, \quad (101)$$

for given initial conditions $u(x, t_0)$ and $h(x, t_0)$. It is clear by regarding this formula that decorrelation at the deterministic level will happen for $\gamma > 1/2$. It is still necessary to find out if at the stochastic level this threshold will be moved to $\gamma > 2/3$. If we consider the dilution KPZ equation (97) then transforming Hopf-Cole we would find the nonlinear equation

$$\partial_t u = \nu \left(\frac{t_0}{t} \right)^{2\gamma} \nabla^2 u - \frac{d\gamma}{t} u \ln(u) + \frac{\gamma F \lambda}{2\nu} t^{\gamma-1} u + \frac{\lambda}{2\nu} \left(\frac{t_0}{t} \right)^{d\gamma/2} \xi(x, t) u, \quad (102)$$

which may be thought of as a time dependent and spatially distributed version of the Gompertz differential equation [31]. In this case it is not evident how to find an explicit solution at the deterministic level and what would be its decorrelation threshold.

Galilean invariance means that the transformation

$$x \rightarrow x - \lambda vt, \quad h \rightarrow h + vx, \quad F \rightarrow F - \frac{\lambda}{2}v^2, \quad (103)$$

where v is an arbitrary constant vector field, leaves the KPZ equation invariant. In case of no dilution this transformation can be replaced by

$$x \rightarrow x - \frac{\lambda}{1-2\gamma}vt_0^{2\gamma}t^{1-2\gamma}, \quad h \rightarrow h + vx, \quad F \rightarrow F - \frac{\lambda}{2\gamma}v^2t_0^{2\gamma}t^{1-3\gamma}, \quad (104)$$

which leaves invariant equation (98). If we consider dilution, then it is not clear how to extend this transformation to leave equation (97) invariant. The main difficulty comes from the dilution term which yields a non-homogeneous contribution to the dynamics as a response to the rotation $h \rightarrow h + vx$. So in summary we may talk of a certain sort of Galilean invariance which is obeyed by the no-dilution KPZ dynamics (98) and is lost when dilution is taken into account. If it were found that the dilution equation (97) obeys the traditional KPZ scaling (at least in some suitable limit), then that would mean the possible necessity for readdressing the role that the symmetries of the KPZ equation have in fixing the universality class [32, 33, 34].

There is still another fundamental symmetry of the KPZ equation, but this time it just manifests itself in one spatial dimension: the so called fluctuation-dissipation theorem [1]. It basically says that for long times, when saturation has already been achieved, the non-linearity ceases to be operative and the resulting interface profile would be statistically indistinguishable from that created by the EW equation. For fast domain growth, we know from the linear theory that the interface never becomes correlated, and it operates, in this sense, as if it were effectively in the short time regime for all times [16]. As a consequence, the fluctuation-dissipation theorem is not expected to play any role in this case. Of course, this result would be independent of whether we contemplated dilution or not.

In more general terms, it is known that the different symmetries of statistical mechanical models influence their scaling properties [35, 36]. It would be interesting to understand in complete generality the interplay among the symmetries of a physical model in a static domain and the asymmetric presence of dilution when we let this domain grow in time.

VIII. CENTER OF MASS FLUCTUATIONS

Another property that has been studied in the context of radial growth, particularly in Eden clusters, is the center of mass fluctuations. It was found numerically that the Eden center of mass fluctuates according to the power law $C_m \sim t^{2/5}$ in $d = 1 + 1$ [4], while in $d = 2 + 1$ there is a strong decrease in this exponent [37]. This reduced stochastic behavior in higher dimensions was already predicted in [14] using radial growth equations, and we will further examine herein the compatibility among the equations and the Eden cluster dynamics. The center of mass fluctuations are characteristic not only of radial growth but also of planar situations. Let us recall the classical EW equation

$$\partial_t h = D \nabla^2 h + \xi(x, t), \quad (105)$$

defined on a one dimensional domain of linear size L_0 and with no flux boundary conditions. It is straightforward to find that the center of mass $h_0(t) = L_0^{-1} \int_0^{L_0} h(x, t) dx$ is a Gaussian stochastic process defined by its two first moments

$$\langle h_0(t) \rangle = 0, \quad \langle h_0(t) h_0(s) \rangle = \frac{\epsilon}{L_0} \min(t, s), \quad (106)$$

and so we have found that the center of mass performs Brownian motion, or equivalently we would say that its position is given by a Wiener process. Note that the fluctuations amplitude decreases with the linear system size, suggesting that in the case of a growing domain our current law $C_m = \langle h_0^2 \rangle^{1/2} \sim t^{1/2}$ will be replaced by a different power law with a smaller exponent. It is easy to see that this result does not hold uniquely for the one dimensional EW equation; indeed, for any d -dimensional growth equation with a conserved growth mechanism, be it linear as the EW or Mullins-Herring equations [1] or nonlinear as the Villain-Lai-Das Sarma equation [38, 39] or its Monge-Ampère variation [23], the center of mass performs Brownian motion characterized by the correlators

$$\langle h_0(t) \rangle = 0, \quad \langle h_0(t) h_0(s) \rangle = \frac{\epsilon}{L_0^d} \min(t, s), \quad (107)$$

as a consequence of the decoupling of the zeroth mode with respect to the surface fluctuations [38]. Note that in the case of non-conserved growth dynamics this is not the case, as illustrated by the KPZ equation

$$\partial_t h = \nu \nabla^2 h + \lambda (\nabla h)^2 + \xi(x, t). \quad (108)$$

It is easy to see that in this case

$$\frac{dh_0}{dt} = \frac{\lambda}{L^d} \int (\nabla h)^2 dx + \xi_0(t) \geq \xi_0(t), \quad (109)$$

where $\xi_0(t) = L^{-d} \int \xi(x, t) dx$ and the equal sign is attained only for $h = \text{constant}$, an unstable configuration for KPZ dynamics. And so one expects stronger center of mass fluctuations in this case. Actually, the short time center of mass fluctuations can be easily calculated for any model which obeys the Family-Vicsek scaling, including the KPZ equation. Indeed, the Family-Vicsek scaling implies the following form of the height-height correlation

$$\langle h(x, t) h(x', t) \rangle = t^{2\beta} C \left(\frac{|x - x'|}{t^{1/z}} \right), \quad (110)$$

which in the short time limit tends to

$$\langle h(x, t) h(x', t) \rangle \sim t^{2\beta+d/z} \delta(x - x'), \quad (111)$$

leading to the result

$$\langle h_0(t)^2 \rangle \sim L^{-d} t^{2\beta+d/z}. \quad (112)$$

And so, within the Family-Vicsek scaling framework, the exponent characterizing the short time behavior of the center of mass fluctuations is $\beta + d/(2z)$.

As we have seen, the center of mass fluctuations are given by the zeroth mode. In the growing domain case it can be shown that the equation controlling the evolution of h_0 is [16]

$$\frac{dh_0}{dt} = -\frac{d\gamma}{t} h_0 + \gamma F t^{\gamma-1} + \left(\frac{t_0}{t} \right)^{d\gamma/2} \xi_0(t), \quad (113)$$

in case dilution is taken into account. In this case we find for long times the center of mass fluctuations

$$C_m^2 = \langle h_0(t)^2 \rangle - \langle h_0(t) \rangle^2 = \frac{\epsilon t_0^{d\gamma}}{L_0^d (d\gamma + 1)} t^{1-d\gamma}, \quad (114)$$

and so $C_m \sim t^{(1-d\gamma)/2}$. If we did not consider dilution we would find in the long time limit

$$C_m^2 = \begin{cases} \frac{\epsilon t_0^{d\gamma}}{L_0^d (1-d\gamma)} t^{1-d\gamma} & \text{if } \gamma < 1/d, \\ \frac{\epsilon t_0}{L_0^d} \ln(t) & \text{if } \gamma = 1/d, \\ \frac{\epsilon t_0}{L_0^d (d\gamma-1)} & \text{if } \gamma > 1/d. \end{cases} \quad (115)$$

If we adapt result (112) to the present setting we find

$$\langle h(x, t) h(x', t) \rangle \sim t^{2\beta+d/z} \delta[t^\gamma(x - x')] = t^{2\beta+d/z-\gamma d} \delta(x - x'). \quad (116)$$

For linear systems the equality $2\beta + d/z = 1$ holds, and so this last equation agrees with (114) but not with (115). This is a consequence of the violation of the Family-Vicsek scaling in the absence of dilution [16, 18]. In the case of the $(1 + 1)$ –dimensional Eden model $d = \gamma = 1$, and if it belonged to the KPZ universality class the center of mass would fluctuate according to the law $C_m \sim t^{1/6}$. This of course does not agree with the measured behavior $C_m \sim t^{2/5}$. This exponent could be recovered by introducing an *ad hoc* instability mechanism, such as for instance considering a growth equation whose zeroth moment obeyed

$$\frac{dh_0}{dt} = D \left(\frac{t_0}{t} \right)^{\delta\gamma} h_0 + \gamma F t^{\gamma-1} + \left(\frac{t_0}{t} \right)^{d\gamma/2} \xi_0(t). \quad (117)$$

The desired exponent is obtained for $\delta = 1$ and $Dt_0 = 2/5$, but however this result is uniform on the spatial dimension and so can not predict the $(2 + 1)$ –dimensional behavior [37]. Additionally this instability mechanism seems to be not enough justified and too non-generic to be a good explanation of the observed phenomenology. Everything points to the fact that the center of mass fluctuations of the Eden model result from a strong violation of the Family-Vicsek ansatz. As we may see from equation (115), this sort of violations imply stronger center of mass fluctuations. This point will be further discussed in the next section.

In summary we can say that the result $C_m \sim t^{2/5}$ suggests a strong violation of the Family-Vicsek scaling by the surface fluctuations of the $(1 + 1)$ –dimensional Eden model. Although the linear law $C_m \sim t^{(1-\gamma d)/2}$ does not reproduce quantitatively the results, we still expect from it a qualitative description of the dynamics, as the strong decrease of this exponent was already reported in $(2+1)$ –dimensions. According to the linear law, the center of mass fluctuations should decrease for increasing growth velocity and spatial dimension. Note also that the nonlinearity seems to be a necessary ingredient; the linearization of the KPZ equation proposed in [11] reads in Fourier space

$$\frac{d}{dt} \langle h_n^2 \rangle = -A|n|^{3/2} \langle h_n^2 \rangle + \frac{B}{|n|^{1/2}}, \quad (118)$$

for some constants A and B and in case of a non-growing domain. This equation supports unbounded fluctuations as revealed by the divergent diffusion in the limit $n \rightarrow 0$, and so this does not constitute a good model for predicting the center of mass fluctuations.

IX. APPLICATIONS TO THE EDEN MODEL

In statistical mechanics it has been customary to classify the behavior of discrete models within universality classes defined by continuum field theories. Non-equilibrium growth theories have been by no means an exception to this rule [40, 41]. In this sense one would be interested in finding the universality class the Eden model belongs to. According to the simulations performed in the planar geometry the Eden model belongs to the KPZ universality class [1]. This agrees with the measured exponent $\beta = 1/3$ in radial systems [4]. However, as we have already seen, there are at least two possible universality classes associated to the KPZ equation in radial systems: dilution-KPZ and dilation-KPZ. The first one is characterized by a behavior more akin to that of planar systems, and the second one by memory effects which imply the departure from the Family-Vicsek ansatz. According to the measurement of the autocorrelation exponent of the Eden model in [11] that yielded $\lambda = 1/3$, *the Eden model would be in the dilation-KPZ universality class* (one would expect $\lambda = 4/3$ for dilution dynamics according to the theory developed herein and in [16]). This fact admits a simple explanation. In the Eden model, cells are aggregated to the colony peripheral in such a way that the positions of already present cells are not modified. Consequently, as the system grows, no dilution is redistributing its constituents. So the rigidity of the Eden model may well be at the origin of the memory effects present at its interface [11], which presumably place it in the dilation-KPZ universality class. But to be sure one would still need, of course, to verify that this implies no contradiction with the center of mass fluctuations as discussed in the last section.

As we have already mentioned, the Eden model may be thought of as an idealization of a developing cell colony. Of course, as it was completely clear from the very beginning [2, 3], there are multiple factors of biological, chemical and even physical nature that are not captured by this model. Apart from them, one could be interested in improving the model in pure statistical mechanical terms. To this end one may look for inspiration in real cell colonies. The structure of a rapidly developing cell colony would be dominated by dilution effects, originated in the birth of new cells which volume causes the displacement of the existent cells. This feature is not captured by any sort of Eden model (diverse proliferation rules, on/off lattice,...) and is fundamental in preserving the Family-Vicsek scaling, as we have already seen. So it seems quite reasonable to modify the Eden model in order to

remove its rigidity, allowing bulk cells proliferation and the displacement of the existent cells, both at the bulk and interface, by the newborn cells. This would not be interesting just in modelling terms, but also for introducing dilution in the model and consequently shifting its universality class.

X. CONCLUSIONS AND OUTLOOK

In this work we have investigated the role of dilution and decorrelation on radial growth. Dilution drives matter redistribution along the growing interface: as the surface becomes larger the already deposited matter occupies a smaller fraction of interface, which is being simultaneously complemented with incoming matter, the actual driving force of domain growth in radial systems. Dilution is important for any rate of domain growth, as it keeps the interfacial density constant, but specially for rapidly growing domains, for which the diffusion mechanism becomes irrelevant and dilution becomes the sole responsible for the propagation of correlations on the macroscopic scale. The importance of dilution is such that in its absence, which takes place in the alternative dilation dynamics, strong memory effects arise. These include an enhanced stochasticity, which separates the behavior of the large spatial scale limit of the two-points correlation function from that dictated by the Family-Vicsek ansatz, and the appearance of non-universal critical exponents in the marginally rough regime, characterized by the equality $\zeta = d$. As have seen, both universality and the Family-Vicsek structure of the correlation function are recovered by virtue of dilution.

As dilution propagates correlations at the same speed at which the interface grows a global correlation becomes impossible for fast domain growth. This leads to decorrelation, or in other words, to a whitening of the interfacial profile in the sense that distant points become uncorrelated. Decorrelation might be macroscopic, which is evident only if we regard the dynamics from a spatial scale of the same order of magnitude of the system size, or microscopic, in which case it is apparent for much smaller length scales. Microscopic decorrelation supports rapid roughening, i. e., growing regimes characterized by $\beta > 1/2$. These appear naturally in the context of radial growth, for instance by considering the IS equation, which results from a geometric variational principle and for which $\zeta = d = 2$ and $\delta = 3$, and thus it shows rapid roughening for all $\gamma > 1$. A consequence of macroscopic decorrelation is the advent of a scale dependent interfacial fractal dimension, which renders

the surface polyfractal and we have conjectured it to be self-similar.

There are several theoretical problems that can be straightforwardly analyzed with the techniques introduced here. We have for instance considered radial interfaces whose mean radius grows as a power law of time $\langle r \rangle \sim t^\gamma$. This result has been obtained by means of a linear mechanism in which an explicit power law dependence on time has been considered, see Eq. (10). This linear mechanism can be substituted by a nonlinear one in which time does not appear explicitly

$$\partial_t r = \gamma F^{1/\gamma} r^{1-1/\gamma} + \frac{1}{r^{d/2} J(\vec{\theta})^{1/2}} \xi(\theta, t), \quad (119)$$

which yields at the deterministic order $R = Ft^\gamma$ again, but it is the source at the first stochastic order of a term (reminiscent of dilution) which may be either stabilizing or destabilizing depending on the value of γ

$$\partial_t \rho = \frac{\gamma - 1}{t} \rho + \frac{1}{F^{d/2} t^{\gamma d/2} J(\vec{\theta})^{1/2}} \eta(\theta, t); \quad (120)$$

for small values of γ the previous sections results are recovered, while for large values of γ memory effects and enhanced (power law) stochasticity appear (which are standard effects of instability as we have already seen), with the threshold value of γ depending of whether we introduce dilution or not (in this concrete example dilution completely erases instability). Also, this instability mechanism, contrary to the ones studied herein and in [15] which rend the zeroth mode unstable and the $l = 1$ ones marginal, is able to destabilize all modes. Different nonlinearities which might destabilize a fixed number of modes lying before some given $l^* \in \mathbb{N}$ can be easily devised too (basically by introducing terms of the form $-r^{-m}$ for some suitable $m \in \mathbb{N}$ in the corresponding equation of motion) and can even be cast on some geometric variational formulation as the cases considered in [15]. Of course, deciding which model is the good one must rely on numerical or experimental evidence based on the study of specific models or systems of interest.

As mentioned in the introduction, part of the motivation for studying radial growth models such as the Eden or different ones lies in the possible similarity of these with some forms of biological development, such as for instance cell colonies. The results of our study can be translated into this context to obtain some simple conclusions, provided the modelling assumptions make sense for some biological system. The structure of a rapidly developing cell colony would be dominated by dilution effects, originated in the birth of new cells which

volume causes the displacement of the existent cells. If the rate of growth is large enough this motion will dominate over any possible random dispersal of the individual cells. It is remarkable that such a consequence simply appears by considering domain growth, while it is not necessary to introduce corrections coming from the finite size of the constituents. This is the dilution dominated situation we have formalized by means of the (decorrelation) inequality $\gamma > 1/\zeta$ (assuming in this case $\delta = \zeta$). If we were to introduce some control protocol in order to keep the consequences of bacterial development to a minimum we would need to eliminate colony constituents (possibly randomly selected) at a high enough rate so the effective growth velocity were one that reversed the decorrelation inequality. For the one dimensional Eden model, accepting it belongs to the KPZ universality class, one finds $\gamma = 1$ and $z = 3/2$. If z played the same role for the nonlinear KPZ equation as ζ for the linear equations considered herein (as it is reasonable to expect), the Eden model would be in the uncorrelated regime. In order to control it we would need to eliminate its cells at rate such that the effective growth rate obeyed $\gamma < 2/3$. For the two dimensional Eden model, if its behavior were still analogous to that of the KPZ equation, we would find $z > 3/2$ and thus a greater difficulty for control. Note that for the particular growth rules of the Eden model one would need to eliminate peripheral cells in order to control the system. This would not be so in the case of an actual bacterial colony, for which bulk cells are still able to reproduce, and so cell elimination could be performed randomly across the whole colony. Of course, these conclusions are speculative as long as radial growth equations are not proved to reasonably model some biological system.

In more general terms, we have found that the surface fluctuations of the Eden model presumably strongly violate the Family-Vicsek scaling. We have identified the absence of dilution in this model as the reason underlying such a violation. In this sense, this model would not be able to describe growing cell colonies, precisely because it assumes a spurious rigidity of bulk cells. On the other hand, it would be better suited to describe the radial growth of crystalline structures [42]. We have also found that reparametrization invariance [20] implicitly implies dilation dynamics. Our results call for an extension of the generalization of Langevin dynamics to arbitrary geometries in order to capture both dilution and dilation scenarios, and the associated bifurcation of universality classes. This same remark would affect as well equilibrium systems, but in this case of course the domain evolution will drive them out of equilibrium, unless growth is quasistatic [43].

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APPENDIX A: HIGHER ORDER PERTURBATION EXPANSION

As we have mentioned in Sec. II, the first order correction in the small noise expansion is a Gaussian stochastic process. We will try to go beyond this order in this appendix, and we will show the difficulties that arise in trying so. We focus again on the radial random deposition equation (10) and assume the solution form

$$r(\vec{\theta}, t) = R(t) + \sqrt{\epsilon}\rho(\vec{\theta}, t) + \epsilon\rho_2(\vec{\theta}, t), \quad (\text{A1})$$

where the noise intensity ϵ will be used as the small parameter [21]. Substituting this solution form into Eq. (10) we obtain the equations hierarchy

$$\partial_t R = F\gamma t^{\gamma-1}, \quad (\text{A2})$$

$$\partial_t \rho_1 = \frac{1}{F^{d/2} t^{\gamma d/2}} \frac{\eta(\vec{\theta}, t)}{J(\vec{\theta})^{1/2}}, \quad (\text{A3})$$

$$\partial_t \rho_2 = -\frac{d}{2F^{1+d/2}} \frac{\rho_1}{t^{\gamma+d\gamma/2}} \frac{\eta(\vec{\theta}, t)}{J(\vec{\theta})}, \quad (\text{A4})$$

where $\xi = \sqrt{\epsilon}\eta$ and both η and ξ are now zero mean quasiwhite Gaussian processes whose correlations are given by

$$\langle \eta(\vec{\theta}, t) \eta(\vec{\theta}', t) \rangle = C(\vec{\theta} - \vec{\theta}') \delta(t - t'), \quad \langle \xi(\vec{\theta}, t) \xi(\vec{\theta}', t) \rangle = \epsilon C(\vec{\theta} - \vec{\theta}') \delta(t - t'), \quad (\text{A5})$$

where $C(\cdot)$ is some regular function approximating the Dirac delta; the necessity for the quasiwhite assumption will be clear in few lines. These equations have been derived assuming $\sqrt{\epsilon} \ll F t^\gamma$, and we will further assume a zero value for both initial perturbations as in Sec. II. The solution to the first two was characterized in Sec. II, where the approximating function $C(\cdot)$ was substituted by the Dirac delta. Here R is a deterministic function and ρ_1 is a zero mean Gaussian stochastic process that is completely determined by its correlation function. The stochastic function ρ_2 is a zero mean process too, but it is not Gaussian this

time, and its correlation (which no longer completely determines the process) is given by

$$\begin{aligned} \left\langle \rho_2(\vec{\theta}, t) \rho_2(\vec{\theta}', s) \right\rangle &= \frac{d^2}{4F^{2+2d}(1-\gamma d)} \times \\ &\left[\frac{(\min\{t, s\})^{2-2\gamma-2\gamma d} - t_0^{2-2\gamma-2\gamma d}}{2-2\gamma-2\gamma d} - t_0^{1-\gamma d} \frac{(\min\{t, s\})^{1-2\gamma-\gamma d} - t_0^{1-2\gamma-\gamma d}}{1-2\gamma-\gamma d} \right] \frac{C(\vec{\theta} - \vec{\theta}')^2}{J(\vec{\theta})J(\vec{\theta}')}, \end{aligned} \quad (\text{A6})$$

if $\gamma d \neq 1$, $\gamma(1+d) \neq 1$, and $\gamma(2+d) \neq 1$. If $\gamma d = 1$ we find

$$\left\langle \rho_2(\vec{\theta}, t) \rho_2(\vec{\theta}', s) \right\rangle = \frac{1}{16F^{2+2d}\gamma^4} \left\{ t_0^{-2\gamma} - [\min\{t, s\}]^{-2\gamma} \left[1 + 2\gamma \ln \left(\frac{\min\{t, s\}}{t_0} \right) \right] \right\} \frac{C(\vec{\theta} - \vec{\theta}')^2}{J(\vec{\theta})J(\vec{\theta}')}, \quad (\text{A7})$$

if $\gamma(1+d) = 1$ then

$$\left\langle \rho_2(\vec{\theta}, t) \rho_2(\vec{\theta}', s) \right\rangle = \frac{d^2}{4F^{2+2d}\gamma} \left[\ln \left(\frac{\min\{t, s\}}{t_0} \right) + \frac{t_0^\gamma}{\gamma} ([\min\{t, s\}]^{-\gamma} - t_0^{-\gamma}) \right] \frac{C(\vec{\theta} - \vec{\theta}')^2}{J(\vec{\theta})J(\vec{\theta}')}, \quad (\text{A8})$$

and if $\gamma(2+d) = 1$ we get

$$\left\langle \rho_2(\vec{\theta}, t) \rho_2(\vec{\theta}', s) \right\rangle = \frac{d^2}{8F^{2+2d}\gamma} \left[\frac{(\min\{t, s\})^{2\gamma} - t_0^{2\gamma}}{2\gamma} - t_0^{2\gamma} \ln \left(\frac{\min\{t, s\}}{t_0} \right) \right] \frac{C(\vec{\theta} - \vec{\theta}')^2}{J(\vec{\theta})J(\vec{\theta}')}. \quad (\text{A9})$$

The long time behavior of the correlations, given by the condition $t, s \gg t_0$, is specified by the following two-times and one-time functions

$$\left\langle \rho_2(\vec{\theta}, t) \rho_2(\vec{\theta}', s) \right\rangle = \frac{d^2}{4F^{2+2d}(1-\gamma d)} \frac{(\min\{t, s\})^{2-2\gamma-2\gamma d}}{2-2\gamma-2\gamma d} \frac{C(\vec{\theta} - \vec{\theta}')^2}{J(\vec{\theta})J(\vec{\theta}')}, \quad (\text{A10})$$

$$\left\langle \rho_2(\vec{\theta}, t) \rho_2(\vec{\theta}', t) \right\rangle = \frac{d^2}{4F^{2+2d}(1-\gamma d)} \frac{t^{2-2\gamma-2\gamma d}}{2-2\gamma-2\gamma d} \frac{C(\vec{\theta} - \vec{\theta}')^2}{J(\vec{\theta})J(\vec{\theta}')}, \quad (\text{A11})$$

when $\gamma(d+1) < 1$, and if $\gamma(d+1) = 1$ then

$$\left\langle \rho_2(\vec{\theta}, t) \rho_2(\vec{\theta}', s) \right\rangle = \frac{d^2}{4F^{2+2d}\gamma} \ln(\min\{t, s\}) \frac{C(\vec{\theta} - \vec{\theta}')^2}{J(\vec{\theta})J(\vec{\theta}')}, \quad (\text{A12})$$

$$\left\langle \rho_2(\vec{\theta}, t) \rho_2(\vec{\theta}', t) \right\rangle = \frac{d^2}{4F^{2+2d}\gamma} \ln(t) \frac{C(\vec{\theta} - \vec{\theta}')^2}{J(\vec{\theta})J(\vec{\theta}')}, \quad (\text{A13})$$

and finally, when $\gamma(d+1) > 1$, we find

$$\left\langle \rho_2(\vec{\theta}, t) \rho_2(\vec{\theta}', s) \right\rangle = \frac{d^2}{8F^{2+2d}} \frac{t_0^{2-2\gamma-2\gamma d}}{1 - (3+2d)\gamma + (2+3d+d^2)\gamma^2} \frac{C(\vec{\theta} - \vec{\theta}')^2}{J(\vec{\theta})J(\vec{\theta}')}, \quad (\text{A14})$$

a correlation function that vanishes in the limit $t_0 \rightarrow \infty$. Now it is clear why we needed the quasiwhite approximation: for a regular function $C(\cdot)$ the expression $C(\cdot)^2$ makes sense,

contrary to what happens if we substitute it by the Dirac delta to get $\delta(\cdot)^2$. This is the first indication of the failure of the higher order perturbation theory.

We now examine the effect that dilution has on the random function ρ_2 , which in this case obeys the equation

$$\partial_t \rho_2 = -\frac{\gamma d}{t} \rho_2 - \frac{d}{2} \frac{(d+1)^{1+d/2}}{F^{1+d/2} t^{\gamma+d/2}} \frac{\rho_1(\vec{\theta}, t) \xi(\vec{\theta}, t)}{J(\vec{\theta})}. \quad (\text{A15})$$

In this case the long time correlation function reads

$$\langle \rho_2(\vec{\theta}, t) \rho_2(\vec{\theta}', s) \rangle = \frac{d^2(d+1)^{2+2d}}{8F^{2+2d}(\gamma d+1)(1-\gamma)} (ts)^{-\gamma d} \min\{t, s\}^{2-2\gamma} \frac{C(\vec{\theta} - \vec{\theta}')^2}{J(\vec{\theta})J(\vec{\theta}')}, \quad (\text{A16})$$

if $\gamma < 1$,

$$\langle \rho_2(\vec{\theta}, t) \rho_2(\vec{\theta}', s) \rangle = \frac{d^2(d+1)^{1+2d}}{4F^{2+2d}} (ts)^{-d} \ln[\min\{t, s\}] \frac{C(\vec{\theta} - \vec{\theta}')^2}{J(\vec{\theta})J(\vec{\theta}')}, \quad (\text{A17})$$

if $\gamma = 1$,

$$\langle \rho_2(\vec{\theta}, t) \rho_2(\vec{\theta}', s) \rangle = \frac{d^2(d+1)^{2+2d}}{8F^{2+2d}(\gamma d+1)(\gamma-1)} (ts)^{-\gamma d} t_0^{2-2\gamma} \frac{C(\vec{\theta} - \vec{\theta}')^2}{J(\vec{\theta})J(\vec{\theta}')}, \quad (\text{A18})$$

if $\gamma > 1$. The one time correlation function is then

$$\langle \rho_2(\vec{\ell}, t) \rho_2(\vec{\ell}', t) \rangle = \frac{d^2(d+1)^{2+2d}}{8F^{2+2d}(\gamma d+1)(1-\gamma)} t^{2-2\gamma} \frac{C(\vec{\ell} - \vec{\ell}')^2}{J(t^{-\gamma} \vec{\ell})J(t^{-\gamma} \vec{\ell}')}, \quad (\text{A19})$$

if $\gamma < 1$,

$$\langle \rho_2(\vec{\ell}, t) \rho_2(\vec{\ell}', t) \rangle = \frac{d^2(d+1)^{1+2d}}{4F^{2+2d}} \ln(t) \frac{C(\vec{\ell} - \vec{\ell}')^2}{J(t^{-\gamma} \vec{\ell})J(t^{-\gamma} \vec{\ell}')}, \quad (\text{A20})$$

if $\gamma = 1$,

$$\langle \rho_2(\vec{\ell}, t) \rho_2(\vec{\ell}', t) \rangle = \frac{d^2(d+1)^{2+2d}}{8F^{2+2d}(\gamma d+1)(\gamma-1)} t_0^{2-2\gamma} \frac{C(\vec{\ell} - \vec{\ell}')^2}{J(t^{-\gamma} \vec{\ell})J(t^{-\gamma} \vec{\ell}')}, \quad (\text{A21})$$

if $\gamma > 1$, where $\vec{\ell} - \vec{\ell}' = t^\gamma(\vec{\theta} - \vec{\theta}')$, $C(\vec{\ell} - \vec{\ell}') = t^{-\gamma d} C(\vec{\theta} - \vec{\theta}')$, and we have assumed that the approximating function $C(\cdot)$ has the same homogeneity as the Dirac delta. Although it is evident that dilution carries out a measurable action, particularly erasing part of the memory effects, the result is far from satisfactory. In all cases the prefactor deviates from the expected random deposition form t^2 [44], the unexpected critical value $\gamma = 1$ has appeared, and for $\gamma \geq 1$ memory effects are present as signaled by the logarithm and the t_0 dependence respectively; and the situation is further complicated by the presence of the factor $C(\cdot)^2$ which becomes singular in the white noise limit. All of these elements suggest the failure of

the small noise expansion beyond the first order. Classical results suggest the possibility of constructing a systematic approach to the solution of some nonlinear stochastic differential equations by continuing the small noise expansion to higher orders [21]. Our present results suggest the failure of this sort of expansions beyond the Gaussian (which turns out to be the first) order in very much the same way as the Kramers-Moyal expansion of the master equation [45] and the Chapman-Enskog expansion of the Boltzmann equation [46] fail beyond the Fokker-Planck and Navier-Stokes orders respectively.

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perturbation theory at this order, rather than an effect related to dilution/dilation.

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